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THE EFFECT OF SINGULARITIES OF THE POTENTIAL ENERGY ON THE INTEGRABILITY OF MECHANICAL SYSTEMS*

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The effect of potential energy singularities on the existence of analytic first integrals of mechanical systems with two degrees of freedom is investigated. Applications to the restricted many-body problem are presented.

1. Formulation of the results. Let M be the configuration manifold of a Lagrangian mechanical system with two degrees of freedom

$$L = T - V + \Lambda \tag{1.1}$$

where L is the Langrangian function, $T = {}^1/_2 \langle q^{\cdot}, q^{\cdot}\rangle$ is the kinetic energy defined on M by the Riemannian metric $\langle , \rangle$, V is the potential energy, and $\Lambda = \langle v(q), q^{\cdot}\rangle$ is a linear function of the velocity defined by the vector field v on M. Bearing in mind applications to celestial mechanics, we will assume that T and Λ are functions of class C^2 on M, and V is a function of class C^2 everywhere on M, except the finite set Σ of singular points of Newtonian type. The point $p \in \Sigma \subset M$ is called a singular point of potential energy of Newtonian type, if in local coordinates q on M with origin at the point p conformal relative to the metric $\langle , \rangle$

$$V = -f(q)/|q|$$

where f is a function of class C^2 and $f(0) > 0$.

We will show that the presence of $n > 2\chi(M)$ singular points of potential energy of the Newtonian type hinders the integrability of a mechanical system. Here $\chi(M)$ is the Euler characteristic of M. The condition $n > 2\chi(M)$ is not satisfied only when M is a sphere $n \leqslant 4$, a plane or projective plane $n \leqslant 2$; M can be a torus a cylinder, a Klein bottle, or a Möbius strip $n = 0$.

We will now give exact formulations. The conformal coordinates on M determine the structure of an analytic manifold on it. Let $H = T + V$ be the energy integral in the phase space $T(M \setminus \Sigma)$. Then, when $h > \sup_M V$ the hypersurface of class C^2

$$\{H = h\} \subset T(M \setminus \Sigma) \tag{1.2}$$

has the natural structure of an analytic manifold. The function is analytic on the hypersurface (1.2), if it can be continued to a function that is analytic in the neighbourhood of the hypersurface (1.2) in $T(M \setminus \Sigma)$ and is uniform with respect to velocity.

Theorem 1. Suppose M is compact and the potential energy V has $n > 2\chi(M)$ singular points of Newtonian type. Then, when

$$h > \sup_{q \in M} H(q, v(q)) = \sup_{q \in M} \left(\frac{1}{2}\|v(q)\|^2 + V(q)\right) \tag{1.3}$$

there are no non-constant analytic functions on the hypersurface (1.2) that are the first integrals of the mechanical system.

Note that the equations of motion of a mechanical system depend only on the differential $d\Lambda$ of the linear form Λ. Hence it is possible to replace in condition (1.3) v by $v + \mathrm{grad}\, f$, where f is any smooth function on M. If V has no singular points, $\chi(M) < 0$ and the mechanical system is reversible, then Theorem 1 is identical with Kozlov's theorem /1/.

When M is non-compact, additional conditions at infinity are necessary. Let $\chi(M) > -\infty$. It is then possible to convert M into a compact two-dimensional manifold $\bar{M}$ by adding a finite number of infinitely distant points ∞_i. Let $D_i \subset M$ be diffeomorphic to disks in the neighbourhood of the points ∞_i. Let us assume that each closed curve in D_i containing the point ∞_i, cannot be extended to infinity in D_i in the class of curves of length bounded in metric $\langle , \rangle$.

The following theorem is the main result of this paper.

Theorem 2. Suppose M be non-compact, the kinetic energy T satisfies the condition at infinity, and the potential energy V has $n > 2\chi(M)$ singular points of Newtonian type. We

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will assume that the differential form $d\Lambda$ maintains its sign. Then by condition (1.3) there will be no non-constant analytic functions at the energy level (1.2) that are the first integrals of the mechanical system.

By definition, the differential form $d\Lambda$ maintains its sign, if either $d\Lambda \equiv 0$ (the mechanical system is reversible) or M is orientable and $d\Lambda = f\Omega$, where $f \geqslant 0$, and Ω is a differentiable 2-form on M that does not vanish.

Example. The restricted cyclic many-body problem. Let n points $p_1, \ldots, p_n$ be fixed in a plane M that rotates around the point $0 \subset M$ at constant angular velocity $\omega \perp M$, and the point $q \in M$ moves under the action of the gravitational attraction of points $p_1, \ldots, p_n$. The Lagrangian function has the form (1.1) where

$$T = {}^1/_2|q^{\cdot}|^2, \quad \Lambda = \langle v(q), q^{\cdot}\rangle = \langle [\omega, q], q^{\cdot}\rangle$$

$$V = -\sum_{i=1}^{n} \frac{\mu_i}{|q - p_i|} - \frac{\omega^2 |q|^2}{2}, \quad \mu_i > 0$$

and the potential energy V has singularities on the set $\Sigma = \{p_1, \ldots, p_n\}$.

The restricted many-body problem is integrable when $n = 1$ and for all ω (Kepler's problem) and, also, when $n = 2$ and $\omega = 0$ (Euler's problem). We shall show that when $n > 2$ and for all ω the first integral of the restricted many-body problem is a function of the Jacobi integral $H = T + V$. We have

$$H(q, v(q)) = V(q) + \frac{|[\omega, q]|^2}{2} = -\sum_{i=1}^{n} \frac{\mu_i}{|q - p_i|} < 0$$

and $d\Lambda = 2|\omega|\Omega$, where Ω is the differential form of an area on the plane M. Since $\chi(M) = 1$, by Theorem 2, when $n > 2$ and $h > 0$, the restricted many-body problem has no analytic non-constant first integrals on the level of the Jacobi integral $\{H = h\}$.

For the restricted three-body problem a similar statement has not been proved. Only weaker theorems are known /2, 3/. There is a hypothesis due to Shazi /4/ on the integrability of the restricted three-body problem on the level of the Jacobi integral $\{H = h\}$, $h > 0$.

We will show that the conditions of Theorems 1 and 2 cannot be weakened.

Theorem 3. Let the set $\Sigma \subset M$, consisting of n points, the kinetic energy T, and $h \in R$ be specified. Then the function $V \leqslant h$ of class C^2 exists on $M \setminus \Sigma$ which has singularities of the Newtonian type on Σ, such that the mechanical system with the Lagrangian function $L = T - V$ has an analytic first integral at the energy level $T + V = h$ and is quadratic with respect to velocity. If M is non-compact or $n \leqslant 2\chi(M)$, then $V < h$.

The proof of Theorems 1 and 2 is based on the existence at the energy level (1.2) of an infinite number of periodic motions with real characteristic indices. We begin by reducing the general case to that in which there are no singularities of potential energy.

2. Regularization. Suppose $n > 2\chi(M)$ is the number of singular points of potential energy.

Lemma 1. A two-dimensional manifold M' and an analytic mapping π: $M' \to M$ exist such that

1) the mapping π: $M' \setminus \pi^{-1}(\Sigma) \to M \setminus \Sigma$ is a covering that is doubly branched over the set Σ;

2) $\chi(M') < 0$.

Proof. We shall consider three cases.

1°. Let n be even. Region $D \subset M$ exist diffeomorphic to a circle in the complex plane C such that $\Sigma \subset D$. We may assume that $D \subset C$. Let f be a polynomial of power n on D that vanishes at points of the set Σ, and let D' be the Riemannian surface of the function $\sqrt{f}$ /5/. Since n is even, the border $\partial D'$ consists of two connected components. We attach to each of them a specimen of $M \setminus D$. The projection π: $M' \to M$ of the manifold obtained on M is a double-sheeted covering doubly branched over Σ. By the Riemann-Hurwitz formula we have $\chi(M') = 2\chi(M) - n < 0$ /5/.

2°. Let $n > 1$ be odd. We select any point $p \in \Sigma$ and construct, as in 1°, a double-sheeted covering π: $N \to M$ branched over $\Sigma \setminus \{p\}$. The set $\pi^{-1}(p)$ consists of two points so that a double-sheeted covering π': $M' \to N$ branching over $\pi^{-1}(p)$ exists. The manifold M' and covering $\pi' \circ \pi$ satisfy the condition of the lemma, and $\chi(M') = 2(2\chi(M) - n) < 0$.

3°. Let $n = 1$. Since $n > 2\chi(M)$, hence $\chi(M) < 0$ so that M is not simply connected. Hence a double-sheeted non-branching connected covering of M exists, and this case reduces to the one already considered.

We fix the value of the energy h and the set $\Sigma' = \pi^{-1}(\Sigma)$.

Lemma 2. A Riemannian metric T', a function V', and a form Λ' linear with respect to velocity of class C^2 on exist M' such that

1) the projection π: $M' \setminus \Sigma' \to M \setminus \Sigma$ that transforms the trajectories of motion of the

mechanical system with the Lagrangian function $L' = T' - V' + \Lambda'$ on TM' of energy $H' = T' + V' = 0$ into trajectories of motion of the initial mechanical system of energy $H = h$;

2) $V' \mid \Sigma' < 0$.

Proof. The Riemannian metric T defines a conformal structure on M. Let T' be an arbitrary Riemannian metric of class C^2 on M' which defines the respective conformal structure on M' and such that $T' = \pi^* T$ outside some neighbourhood of the set Σ'. Since $\pi: M' \to M$ is a conformal mapping, a non-negative function $f \in C^2(M')$ exists that $\pi^* T = fT'$.

We set

$$V' \mid M' \setminus \Sigma' = f(V \circ \pi - h), \quad \Lambda' = \pi^* \Lambda$$

Let $g = 2\sqrt{(h - V)T} + \Lambda$ be the Jacobi metric on $M \setminus \Sigma$ that corresponds to energy h, and let $g' = 2\sqrt{-V'T'} + \Lambda'$ be the Jacobi metric on $M' \setminus \Sigma'$, that corresponds to zero energy. To prove the first statement of the lemma it is sufficient by Maupertius's principle to show that $\pi^* g = g'$ on $M' \setminus \Sigma'$. But this follows from the definition of V' and Λ'.

It remains to prove that for any point $p \in \Sigma'$ the function V' is continuable to a function of class C^2 in the neighbourhood of p and $V'(p) < 0$. Let $q: U \to C$ be the conformal coordinate in the neighbourhood U of the point $\pi(p)$ with its origin at $\pi(p)$, and $\zeta: U' \to C$ the respective Levi-Civita coordinate /6/ in the neighbourhood $U' = \pi^{-1}(U)$ of the point p: $\zeta^2 = q \circ \pi$. Then the Jacobian of the mapping π in coordinates ζ, q is $|2\zeta|^2 = 4|q \circ \pi|$ so that the metric $\pi^* T$ is divisible by $|q \circ \pi|$. Therefore the function f is divided by $|q \circ \pi|$. By definition of the singular point of Newtonian type $V' = fV \circ \pi - fh$ is continuable to a function of class C^2 in region U' and $V'(p) < 0$. The lemma is proved. We shall call the mechanical system with the Lagrangian function $L' = T' - V' + \Lambda'$ a regularized system.

Corollary. Let an analytic first integral of a mechanical system exist at the energy level (1.2). Then the analytic first integral of the regularized system exists at the energy level $\{H' = 0\}$.

Proof. By Lemma 2 a regularized system has an analytic first integral F in region $U = \{H' = 0\} \cap T(M' \setminus \Sigma')$. It remains to show that F is continuable to the analytic function F' on $\{H' = 0\}$. Let g^t be the phase flux of a regularized system. Since $\{V' = 0\} \cap \Sigma' = \phi$, the set $\{H' = 0\} \setminus U$ does not contain the equilibrium position of the flux g^t, and the trajectories of g^t are transversal $\{H' = 0\} \setminus U$. Hence for fairly small $t > 0$ we have $g^t U \cup U = \{H' = 0\}$. We put $F' \mid g^t U = F \circ g^{-t}$. Since $F = F \circ g^{-t}$ on $U \cap g^t U$, hence F' is a correctly defined analytic function on $\{H' = 0\}$, which it was required to prove.

If the input mechanical system satisfies the conditions of Theorems 1 and 2, the regularized system has the same properties. Therefore, when proving Theorems 1 and 2, we may assume that V is a function of class C^2 over the whole M and $\chi(M) < 0$.

3. The first integrals of the geodesic flux. The non-existence of first integrals (Theorems 1 and 2) is derived from the following general statement. Let M be a smooth two-dimensional manifold and $N \subset M$ a two-dimensional submanifold with border ∂N. We call ∂N geodesically convex in the Finslerian metric g on M, if the following conditions are satisfied. Let $t \to \gamma(t)$ be the geodesic metric g such that $\gamma(0) \in \partial N$, and let the vector $\dot\gamma(0)$ be tangent to ∂N. Then $\gamma(t) \in \overline{M \setminus N}$ for fairly small $t \in R$.

Lemma 3. Let g be a positive definite Finslerian metric of class C^2 on the connected two-dimensional analytic manifold M. Let $N \subset M$ be a compact two-dimensional submanifold with a border, the submanifold being such that the border ∂N is geodesically convex in metric g and $\chi(N) < 0$. Then non-constant analytic first integrals of the geodesic flux on $T_1 M$ do not exist.

When $N = M$ and the border ∂N is empty, while g is the Riemannian metric on M, we have Kozlov's theorem /1/. The proof of this lemma is given in /7/. The Riemannian metric was considered there but its extension to a Finslerian metric does not present difficulties.

If the border ∂N is empty, the proof of the lemma repeats the proof of Kozlov's theorem. Let ∂N be non-empty. We assume that the equations of the geodesic flux have an analytic first integral F on $T_1 M$. Since M is non-compact, the Liouville theorem cannot be applied. However, using the convexity of ∂N, we can show that each non-singular surface of the level of integral F, which contains the periodic trajectory of the geodesic flux in $T_1 N$, is a two-dimensional torus contained in $T_1 N$.

Since $\chi(N) < 0$ and the border ∂N is non-empty, the fundamental group $\pi_1(N)$ is a free non-Abelian group. Since the border ∂N is convex, the shortest closed geodesic in N with real characteristic indices correspond to each class of conjugate elements $\pi_1(N)$. Using the property of surfaces of the level F presented above, and the generalization of the theorem /8/ on asymptotic geodesics, it can be shown that the integral F has an infinite number of critical values on $T_1 N$ and is, consequently, constant. We omit the details.

Let the potential energy by $V \in C^2(M)$, $\chi(M) < 0$, let h be the energy, and $g = 2\sqrt{(h - V)T} + \Lambda$ the Finslerian Jacobi's metric on M. To prove Theorem 1 it is sufficient to prove that the following lemma holds.

Lemma 4. Under condition (1.2) the Jacobi metric is positive definite: $g \geqslant 2\sqrt{(h - H(q, v(q)))T}$.

The proof follows from the Cauchy inequality, i.e.

$$|\Lambda| = |\langle v(q), q^{\cdot}\rangle| \leqslant \|v(q)\| \cdot \|q^{\cdot}\|$$

so that

$$g \geqslant 2\sqrt{T}\left(\sqrt{h - V(q)} - \sqrt{{}^1/_2\|v(q)\|^2}\right) \geqslant 2\sqrt{(h - V(q) - {}^1/_2\|v(q)\|^2)T}$$

since $h - V(q) > {}^1/_2\|v(q)\|^2$.

To prove Theorem 2 it is necessary to construct the submanifold $N \subset M$ that satisfies the condition of Lemma 3. Let the conditions of Theorem 2 be satisfied.

Lemma 5. A submanifold $N \subset M$ exists such that N is homotopically equivalent to M, and the border ∂N is geodetically convex in the Jacobi metric.

Proof. Let the mechanical system be irreversible (the case of reversibility is simpler). Then M is orientable, and $d\Lambda = f\Omega, f \geqslant 0$, where Ω is a differentiable 2-form on M which does not vanish. The form Ω defines the orientation of M.

Lemma 6. Let the oriented boundary of region $D \subset M$ be the closed geodesic of the Jacobi metric. Then region $\overline{M \setminus D}$ is geodesically convex in the Jacobi metric.

Proof. Let τ be the vector of the tangent to the oriented curve ∂D at the point $p \in \partial D$, and n the vector of the inner normal to ∂D: $\langle\tau, n\rangle = 0$; $\|\tau\| = \|n\| = 1$. Then by definition of the oriented boundary $\Omega(\tau, n) > 0$. We shall show that the small geodesic γ of metric g which is tangent to ∂D at the point p is entirely contained in D. If γ and ∂D are equally oriented at the point p, then $\gamma \subset \partial D$. Let γ and ∂D be oppositely directed. By Maupertius's principle the curves ∂D and γ are the trajectories of motion $t \to \gamma_+(t)$ and $t \to \gamma_-(t)$ of a mechanical system with energy h and initial conditions

$$\gamma_\pm(0) = p, \quad \gamma_\pm^{\cdot}(0) = v_\pm = \pm\sqrt{2(h - V(p))}\,\tau \tag{3.1}$$

The curvature $k_\pm$ of trajectories $\gamma_\pm$ are determined at the point p for the selected normal at that point. From the equations of motion we have

$$k_\pm\|v_\pm\|^2 = -\langle \mathbf{grad}\, V(p), n\rangle - d\Lambda(v_\pm, n)$$

By virtue of (3.1)

$$k_- - k_+ = \frac{d\Lambda(v_+ - v_-, n)}{\|v_\pm\|^2} = \sqrt{\frac{2}{h - V(p)}}\, f(p)\,\Omega(\tau, n) \geqslant 0$$

Hence the curvature k_- of the curve γ at the point p is greater than the curvature k_+ of the curve ∂D at the same point p, which it was required to prove.

Let $D_i \subset M$ be the neighbourhood of infinitely distant points of M that are diffeomorphic to disks, and ∂D_i be the oriented boundaries of regions D_i. Since M is multiply connected, curves ∂D_i are non-contracting in M. If follows from Lemma 4 that the Jacobi metric g satisfies on M the condition at infinity, which is similar to the condition $\langle,\rangle$ on the Riemannian metric. We derive from here by standard methods of the calculus of variation as a whole /9/ that the homotopic class of every closed and oriented curve ∂D_i contains the shortest closed geodetic Γ_i of the Finslerian metric g.

It is sufficient to show that the geodesics Γ_i are non-selfintersecting and $\Gamma_i \cap \Gamma_j = \varnothing$ when $i \neq j$. Then by Lemma 6 the curves Γ_i bound the submanifold $N \subset M$ that satisfies the condition of Lemma 5.

Since the Jacobi metric g is irreversible, the usual Liusternik-Shirel'man method is inapplicable. It can be shown that the geodesic Γ_i is the oriented boundary ∂C_i of the 2-chain C_i imbedded in $\overline{M}$ with retained orientation.

The following is an outline of the proof. Let $S(\gamma)$ be the length γ in the Jacobi metric g for any oriented curve $\gamma \subset M$. By Lemma 4 $S(\gamma) \geqslant 0$. Let $\gamma_t, t \geqslant 0; \gamma_0 = \partial D_i$ be the homotopy of the closed curve ∂D_i that reduces the action of S /9/. It can be assumed that γ_t is a broken geodesic. Let $\tau > 0$ be the greatest number such that when $t < \tau$ the curve γ_t is the boundary of the 2-chain imbedded in $\overline{M}$ with retained orientation. Then, when $t = \tau$ the touching of two arcs of the curve γ_τ occurs at some point p (the idea of the inner side of curve γ_τ is, by our assumption well-posed). Point p divides γ_τ into two closed curves, one of which is homotopic to γ_τ in M. Denoting it by γ_τ', we have $S(\gamma_\tau') \leqslant S(\gamma_\tau)$ so that it is possible to replace γ_τ by γ_τ'. It is, thus, possible to obtain the homotopy ∂D_i that reduces the action in the class of curves that are the boundary of the 2-chain imbedded in $\overline{M}$ with retained orientation, which it was required to prove.

We shall show that curves Γ_i and Γ_j do not intersect. The same reasoning applied for proving that curve Γ_i is non-selfintersecting. The proof is based on the sign-constancy of the differential form $d\Lambda$ for any 2-chain D imbedded in M with retained orientation.

$$\iint_D d\Lambda \geqslant 0$$

Let $\Gamma_i = \partial C_i$, $\Gamma_j = \partial C_j$ and $\Gamma_i \cap \Gamma_j \neq \varnothing$. Then $D = C_i \cap C_j$ is a 2-chain (possible degenerate) imbedded in M with retained orientation. We have $\partial D = \alpha + \beta$, where $\alpha \subset \Gamma_i$ and $\beta \subset \Gamma_j$ are oriented 1-chains whose orientation is the same as that of Γ_i and Γ_j.

The action of the Jacobi oriented curve $\gamma \subset M$ is

$$S(\gamma) = L(\gamma) + \int_\gamma \Lambda$$

where $L(\gamma)$ is the length of curve γ in the Riemannian metric $2\sqrt{(h-V)\,T}$. We assume that $L(\alpha) \geqslant L(\beta)$ and set $\Gamma = (\Gamma_i \setminus \alpha) - \beta$, where $-\beta$ is the chain β of opposite orientation. Then Γ is a closed curve homotopic to Γ_i, and

$$S(\Gamma_i) - S(\Gamma) = S(\alpha) - S(-\beta) = L(\alpha) + \int_\alpha \Lambda - L(\beta) + \int_\beta \Lambda =$$

$$L(\alpha) - L(\beta) + \iint_D d\Lambda \geqslant 0$$

Since Γ_i is the shortest closed geodesic homotopic to ∂D_i, Γ is also a geodesic of the Jacobi metric /9/, and $S(\Gamma) = S(\Gamma_i)$. Consequently $\Gamma = \Gamma_i$, from which $\alpha = -\beta$. But this contradicts the convexity of curves Γ_i and Γ_j (Lemma 6). Theorem 2 is proved.

4. Integrable systems. Let T be the Riemannian metric of class C^2 on the two-dimensional manifold M (the kinetic energy). We construct the potential energy $V \leqslant h$ on $M \setminus \Sigma$ that satisfies the condition of Theorem 3. Let us assume that either M is non-compact or $n = 2\chi(M)$. The remaining cases are similarly treated.

Lemma 7. A function $V < h$ of class C^2 on $M \setminus \Sigma$ exists that has singularities of Newtonian type on Σ such that $\frac{1}{2}\Delta \ln(h - V) = K$, where K is the Gaussian curvature of metric T, and Δ is the Laplace operator.

Proof. Let U be any smooth positive function on $M \setminus \Sigma$ such that the function $-U$ has singularities of the Newtonian type at points of set Σ. Then in conformal coordinates q in the neighbourhood of the point $p \in \Sigma$ we have $U = f(q)/|q|$, $f(0) > 0$. Hence

$$\ln U(q) = \ln f(q) - \tfrac{1}{2}(\ln q + \ln \bar{q}), \quad \Delta \ln U = \Delta \ln f \tag{4.1}$$

so that $\Delta \ln U$ is a smooth function on M. It is sufficient to show that a smooth function φ exists on M and is such that

$$\Delta\varphi = K - \tfrac{1}{2}\Delta \ln U \tag{4.2}$$

Indeed, it is then possible to set $V = h - Ue^{\varphi}$.

If the manifold M is non-compact, Eq. (4.2) always has a solution /5/. We assume that M is compact, and $n = 2\chi(M)$. For simplicity we assume M to be oriented and Ω to be the differential form of an area in M. By Green's formula and by virtue of (4.1) we have

$$\iint_M \frac{1}{2}\Delta \ln U \cdot \Omega = -\sqrt{-1}\iint_M d(\partial \ln U) = 2\pi \sum_{p\in\Sigma} \operatorname{res}_p \partial \ln U =$$

$$\pi n \cdot \operatorname{res}_0 \frac{dq}{q} = \pi n$$

By the Gauss-Bonnet formula

$$\iint_M \left(K - \frac{1}{2}\Delta \ln U\right)\Omega = 2\pi\chi(M) - \pi n = 0$$

Consequently Eq. (4.2) has a solution. The lemma is proved.

The Gaussian curvature of the Jacobi metric $g = 2\sqrt{(h-V)\,T}$ of the system constructed is

$$K_g = \frac{1}{2(h-V)}\left(K - \frac{1}{2}\Delta \ln(h - V)\right) = 0$$

Hence the Jacobi metric of the system regularized on M' is Euclidean. It can be shown that the regularized system has the first integral F on the zero level of energy in TM' which is linear with respect to velocity. In the compact case M' is a torus, os that this is obvious. Let $\pi: M' \to M$ be a double sheeted covering constructed in Lemma 1, and $\sigma: M' \to M'$ be the involution transposing the layers $\pi \circ \sigma = \pi$ and $\sigma^2 = 1$. It can be shown that $\sigma^* F = -F$. Hence the function F^2 is invariant to involution σ and, consequently, reduces by projection π to the first integral, which is quadratic with respect to velocity, of the system constructed at the level of energy $\{T + V = h\}$.

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ON THE STABILITY OF A SOLID ROTATING AROUND THE VERTICAL AND COLLIDING WITH A HORIZONTAL PLANE*

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The motion of a heavy solid with a convex surface above an absolutely smooth horizontal plane is considered. Collisions of the body with the plane during its motion are assumed to be absolutely elastic. The stability of such motion is investigated when the body rotates at constant angular velocity around the vertical, while its centre of mass moves between collisions on a parabola or along a fixed vertical line coinciding with the axis of rotation of the body. Stability conditions are obtained to a first approximation for arbitrary values of the parameters of the problem. Special cases of a non-rotating body with geometrical and dynamic symmetry, and of a body whose surface in the neighbourhood of the point of contact with the plane is close to spherical, are analyzed in detail. A peculiar "quantification" of stability and instability along the height of jumps of the body over the plane was found in the case of a rotating body.

The problem of the stability of the motion of a solid with a convex surface of arbitrary form and an arbitrary inertia tensor when there is a non-retaining connection, has not so far been investigated. Investigations in the theory of vibrating-collision systems have dealt with either material points or homogeneous spheres, which in the case of a smooth plane is, from the point of view of dynamics, the same.

1. Let a solid move in a gravitational field above a stationary horizontal plane. The surface of the body is assumed to be convex, and the plane is assumed to be absolutely smooth. During its motion the body may touch the plane at a point on its surface. Then, if a collision occurs, it is assumed to be absolutely elastic.

Let $Oxyz$ be a system of coordinates with origin at the point O of the horizontal plane. The Oz axis is directed vertically upward. We denote the coordinates of the centre of mass G by x, y, z, and attach to the body a system of coordinates $G\xi\eta\zeta$ whose axes are directed along its principal central axes of inertia. The orientation of the body relative to absolute space is defined by Euler's angles θ, φ, ψ which are conventionally introduced. We denote the

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